Texas Math Talent Search 2006/07 Problems

Contest rules

1. You must be a Texas resident and a K-12 student to take part in the contest.

2. The problems assume knowledge of high school mathematics.

3. All work must be done independently. Help from parents, friends or teachers is not allowed.

4. Each problem’s solution must be given on one or more separate sheets of paper. Do not write solutions of several problems on the same sheet. Print you name on each sheet of paper.

5. Solve as many problems as you can.

6. Be as complete as you can describing your solutions. Show all work. Justify your claims. Answers without reasons to support them do not count as solutions. Write clearly and legibly.


8. To submit your solutions, write on a single cover sheet or type the following:
   Your name
   Street Address, City, TX Zip code
   Telephone, email address (if available)
   The name of your school and the name of your teacher who will submit your solutions
   Ask your teacher to include his/her information in a similar manner on a separate sheet of paper and to mail your solutions to
   Texas Math Talent Search
   Department of Mathematics
   Texas A& M University
   College Station, TX 77843-3368

9. If you are not officially enrolled in a public, private, or home school, and thus cannot submit your solutions through a teacher, please email the organizers at: talentsearch@math.tamu.edu

10. You will be notified of your score in a timely manner.

11. Students who submit the best solutions will be invited to a special event at the Mathematics Department some time in Spring 2007. During this event a brief problem contest will determine the final winners. Then the prizes will be awarded.

   Good luck with your work!
Six Contest Problems

Problem 1. Find digits that after substituting instead of letters, make the following equality correct:

\[
\begin{array}{c}
\text{forty} \\
+ \ 	ext{ten} \\
\text{ten} \\
\hline 
\text{sixty}
\end{array}
\]

Different letters should correspond to different digits, and same letters to the same digits.

Problem 2. Let \( T \) be a triangle of area 6 square inches and perimeter 12 inches. Show that the triangle \( T \) can be cut into 2006 triangles each of which has perimeter strictly larger than 6 inches.

Problem 3. Let \( a, b, c \) and \( d \) be positive real numbers such that \( a < b < c < d \) and \( ad = bc \). Show that \( b + c < a + d \).

Problem 4. Three points \( A, B \), and \( M \) lie on a circle. The distances from the point \( M \) to the lines tangent to the circle at the points \( A \) and \( B \) are equal to \( a \) and \( b \) respectively. Find the distance from the point \( M \) to the line \( AB \).

Problem 5. Show that for every triangle \( ABC \) the inequality \( \cos A + \cos B + \cos C \leq \frac{3}{2} \) holds, where \( A, B, C \) denote the radian measures of the three angles of the triangle.

Problem 6. Let \( m(x, y) = \frac{2xy}{x+y} \) and \( M(x, y) = \frac{(x+y)}{2} \). Both these formulas represent, in some sense, an average of \( x \) and \( y \). What happens when we average the averages, and then average those, and so on? Specifically: given an ordered pair \( u = (x, y) \), let \( A(u) = (m(x, y), M(x, y)) \), and consider the pairs \( u, A(u), A(A(u)) \) and so on.

1. Show that if \( 0 < x < y \), then \( x < m(x, y) < M(x, y) < y \). In other words, if one draws the pair \( u = (x, y) \) of numbers as a segment of the real axis, the pair \( A(u) = (m(x, y), M(x, y)) \) corresponds to a smaller segment:

\[
\begin{tikzpicture}
  \draw[->] (0,0) -- (2,0) node[midway, above] {\( m(x, y) \)};
  \draw[->] (0,0) -- (0,-2) node[midway, left] {\( u \)};
  \draw[->] (0,0) -- (2,-2) node[midway, right] {\( M(x, y) \)};
  \draw[->] (0,0) -- (0,2) node[midway, above] {\( A(u) \)};
  \draw[dashed] (0,0) -- (2,2) node[midway, right] {\( A(A(u)) \)};
\end{tikzpicture}
\]

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2. Show that there is a single number $M(x, y)$ that lies between the two entries of $u, A(u), A(A(u)), A(A(A(u)))$ and so on, i.e.

$$m(x, y) < m(m(x, y), M(x, y)) < \ldots M(x, y) \ldots < M(m(x, y), M(x, y)) < M(x, y).$$

3. Find and prove a formula for $M(x, y)$. 

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