The Spiral - One of nature’s favorite curves

Spirals, a curve you have drawn so many times, is perfectly suited to the parametric form. Clearly, you can see in the graphics below, a spiral is not represented as a function. Why? There are many types of spirals, and the source functions for defining them are the parent functions. We’ll consider only those two shown below.

Spiral Basics

Circles come first. Let’s review that a circle has the parametric form

\[ x = a \cos t \]
\[ y = a \sin t \]

where usually \( 0 \leq t \leq 2\pi \). If the range on \( t \) is greater than \( 2\pi \), the circle is retraced; if the range on \( t \) is less than \( 2\pi \), only a portion of the circle is expressed. By squaring both equations and adding, we have

\[ x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t \]
\[ = a^2(\cos^2 t + \sin^2 t) \]
\[ = a^2 \]

by the trig identity that \( \cos^2 t + \sin^2 t = 1 \) for all angles \( t \). The alternate form for the circle is then \( x^2 + y^2 = a^2 \). Of course, this is not a function either. However, solving for \( y \) gives two solutions

\[ y = \sqrt{a^2 - x^2} , \]
\[ y = -\sqrt{a^2 - x^2} \]

These are functions; the first is the upper semicircle, and the second is the lower semicircle. One thing is important to note is the parameter \( t \) is actually the angle the point on the circle
makes with the horizontal. In the illustration below we have taken \( a = 2 \).

The parameter is the angle

**Sprials.** Looking carefully at the image of a spiral, we see that it is circular-like, but the radius is ever getting larger. This suggests we should reconsider the parametric equations for the circle and allow the radius to change with the parameter, or angle. We can actually proceed in any way we wish, but let’s stick to the “spirals book” for a while anyway.

**Archimedean form.** Consider the very slight modification of the parametric equations of the circle, where the multiplication of the radius by the parameter is made.

\[
\begin{align*}
x &= at \cos t \\
y &= at \sin t
\end{align*}
\]

In this way, the spiral can be viewed as a circle with ever changing radius: different angle \( \rightarrow \) different radius. This is a good way to view the spiral, because it suggests that we can achieve every changing spiral by how we change this radius. When the “radius” is \( at \), we will note that the radius is changing linearly. So, by multiplying two parent functions (\( t \) and \( \sin t \) or \( \cos t \)) we have the following spiral.
It is called the Archimedean spiral, named after the greatest mathematician of antiquity, Archimedes, who discovered it. He used it in a variety of ways, most particularly to square the circle. (Recall your geometry here.) The Archimedean spiral occurs in nature, most notably as a favorite food, the cinnamon bun.

The derivation for the radius: Follow the steps above. By squaring both equations and adding we have

\[ x^2 + y^2 = (at)^2 \cos^2 t + (at)^2 \sin^2 t \]
\[ = (at)^2 (\cos^2 t + \sin^2 t) \]
\[ = (at)^2 \]

by the trig identity \( \cos^2 t + \sin^2 t = 1 \) for all angles. There really is no alternate form for the spiral that corresponds to the circle, because we cannot eliminate the parameter (aka angle) \( t \).

**Equiangular form.** Instead of multiplying by \( t \), we could have multiplied by the exponential \( e^t \).
Its graph is shown below. To make an interesting graph we used the following parametric equations: $x = a e^{t} \cos t$, $y = a e^{t} \sin t$.

Equiangular spiral

One question that immediately comes to mind is why the moniker “equiangular?" That can be illustrated in the following diagram:

The angle $\phi$ is the same no matter what radial line is selected. This can be proved easily with just a little calculus. This spiral was discovered by Rene Descartes, and its properties of self-reproduction by Jacob Bernoulli (1654-1705) who requested that the curve be engraved upon his tomb with the phrase "Eadem mutata resurgo" ("I shall arise the same, though changed.")

The equiangular spiral seems to be nature’s favorite. Following is a picture gallery of equiangular spirals. The familiar chambered Nautilus shell is in the form of an equiangular...
Many kinds of spirals are known, the first dating from the days of ancient Greece. The curves are observed in nature, and human beings have used them in machines and in ornaments, notably architectural. Other plane spirals are Euler’s, or Cornu’s, or Clothoid; Cotes’, Fermat’s, or parabolic; lituus; Poinsot’s; reciprocal, or hyperbolic; and sinusoidal. Here is Fermat’s spiral.
From right here on planet Earth we see the pea tendril as a spiral. Look carefully at a newly budding plant, and notice how it unfolds in a spiral.

The sunflower shown below also shows a spiral. There is an interesting connection with the Fibonacci sequence here.
Other seashells show their spiral growth.

http://xahlee.org/SpecialPlaneCurves_dir/specialPlaneCurves.html

From the grocery store we have the cauliflower and pineapple.
We can recast the entire discussion using polar coordinates. A spiral is a plane curve that, in general, unwinds around a point while moving ever farther from the point.
While there are many kinds of spirals, two of the most important are the Archimedean spiral and the equiangular spiral. The Archimedean spiral is described in polar coordinates by

\[ r = a\theta \]

It was discovered by Archimedes in about 225 BC in a work *On Spirals*. It has been used to trisect angles and to square the circle. As you can see, the radius increases by a constant amount each revolution.

The equiangular spiral is given by

\[ r = a e^{b\theta} \]

References