Elliptical Orbit Exploration

Materials. You will need the Elliptical Explorer. You may upload the Elliptical Explorer. Be sure to upload into the same folder BOTH the files "elliptical_orbit.html" and "elliptical_orbit.swf."

Elliptical Orbit

Requirements. Students should be familiar with the conic sections. They should also be studying parametric equations.

Introduction. In this activity we explore visually what ellipses look like under translations and expansions. The full parametric form for an ellipse is

\[
\begin{align*}
x &= a + r_1 \cos t \\
y &= b + r_2 \sin t
\end{align*}
\]

Here \((a, b)\) is the center and the \(r_1\) and \(r_2\) are the semi-major and semi-minor axes. An example of an ellipse is graphed below. It is

\[
\begin{align*}
x &= 2 + 3 \cos t \\
y &= -1 + 2 \sin t
\end{align*}
\]

which means the center is at \((2, -1)\) with semi-major axis 3 and semi-minor axis 2.

The Elliptical Orbiter Exploratory simultaneously draws ellipse of any desired configuration but also moves a “planet” about it. You can trace the planet, or stop/start the orbiting. Below we show the layout of the orbiter.
There are several, very self-explanatory controls. First of all there are the boxes pertaining to $a$, $b$, $r_1$, and $r_2$. You can move the slider to adjust these values — OR you can enter precise values in the respective boxes. This will give an ellipse graphed exactly as desired. With the “Auto” buttons, you can have these parameters change dynamically, the effect being an ellipse of changing semi-major axes and/or with a changing center — all the time with the “planet” orbiting about. Next you can stop and start the orbiting using the appropriate buttons, and by pressing the “trace button,” trace the coordinates of the the planet. In the image below the coordinates of the planet are shown above the upper left corner of the graph.
Activities

1. Explore the Elliptical orbiter as an interactive activity. Remember, when \( r_1 = r_2 \) there results a circle.

2. Notice that when the sign of either \( r_1 \) or \( r_2 \) is minus, the direction of the orbiter reverses to become clockwise. Why is this? Notice when both signs are minus, the direction is again counter clockwise. Why is this? It makes for an interesting discussion. We have the table

<table>
<thead>
<tr>
<th>Sign of ( r_1 )</th>
<th>Sign of ( r_2 )</th>
<th>Direction of orbit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 &gt; 0 )</td>
<td>( r_2 &gt; 0 )</td>
<td>clockwise</td>
</tr>
<tr>
<td>( r_1 &lt; 0 )</td>
<td>( r_2 &gt; 0 )</td>
<td>clockwise</td>
</tr>
<tr>
<td>( r_1 &gt; 0 )</td>
<td>( r_2 &lt; 0 )</td>
<td>clockwise</td>
</tr>
<tr>
<td>( r_1 &lt; 0 )</td>
<td>( r_2 &lt; 0 )</td>
<td>counter clockwise</td>
</tr>
</tbody>
</table>

Mathematically speaking, we often speak of the counter clockwise direction as the “positive” direction, and the clockwise direction as the “negative” direction.

3. Notice that the curve is the same in all three cases above. This illustrates that parametric curves are not unique. There can be several representations of the same curve. For example, the standard parabola \( y = x^2 \) has many parametric representations.

\[
\begin{align*}
x &= t & y &= t^2 & -\infty < t < \infty \\
x &= -u & y &= u^2 & -\infty < u < \infty \\
x &= s^3 & y &= s^6 & -\infty < s < \infty
\end{align*}
\]

Can you think of more? (Hint. We need two conditions: What \( x \) must range from \(-\infty \) to \( \infty \). What is \( y \) must be the square of what is \( x \).) For example suppose we take \( x = 3t - 2 \). What is \( y \)? What is the range of \( t \) ?

4. Approximate the area of the ellipse by counting the number of squares strictly inside the ellipse. Now include all the squares that also touch the ellipse. This gives an upper estimate and a lower estimate of the area. We know the area of an ellipse is \( A_{\text{ellipse}} = \pi r_1 r_2 \). Consider the example with \( r_1 = 3 \) and \( r_2 = 5 \). The exact area is
$A_{\text{ellipse}} = \pi r_1 r_2 = \pi(3)(5) = 15\pi \approx 47.124$. Now count the squares

Those completely inside are 28; those touching are 28. (By symmetry, this number must be divisible by 4.) The total of those touching and those inside is 56 — a not-so-good estimate of the area. Now the larger the ellipse the better the approximation.

5. Improvement of the approximation. There are several schemes to improve the accuracy of the area approximation. (1) Assuming that when the ellipse cuts a square, it cuts it in half, we are led to take the total number of squares inside plus half the number of touching squares. Thus our approximation is $28 + \frac{1}{2}28 = 42$. This is a good approximation. (2) We might also average the number of total encapsulating rectangle and the number of squares strictly inside. For the problem above, we have the encapsulating rectangle (in the first quadrant) is $3 \times 5 = 15$. This averaged with the number of squares strictly inside (7) is $\frac{1}{2}(7 + 15) = 11.0$. Now multiply by 4, the number of quadrants. This gives an approximate area of 44. Can you think of other approximating schemes?

6. Make your estimates of area making only half the measurements of horizontal or vertical symmetries. Can you make only a quarter of the counts using horizontal AND vertical symmetries?

7. Approximate the perimeter of the ellipse by counting the number of squares that the ellipse touches. The perimeter of an ellipse is a difficult problem that requires calculus to formulate. Even still, the resulting integral cannot be exactly computed. The perimeter actually is

$$P_{\text{ellipse}} = \int_0^{2\pi} \sqrt{r_1^2 \sin^2 t + r_2^2 \cos^2 t} \, dt$$

For example if $r_1 = 2$ and $r_2 = 1$, the perimeter is

$$P_{\text{ellipse}} = \int_0^{2\pi} \sqrt{4 \sin^2 t + \cos^2 t} \, dt \approx 9.6884$$

This ellipse graphs as

Counting the number of squares the ellipse touches is 8. You will note that this estimate is relatively crude because in most cases the length of the path of the ellipse through any of
the squares is a curve. For the ellipse in Problem 3, the perimeter is

\[ P_{\text{ellipse}} = \int_0^{2\pi} \sqrt{9\sin^2 t + 25\cos^2 t} \, dt \approx 25.527 \]

The estimate given about (those touching), which is 28 is a reasonable but not outstanding approximation.

8. Try these same estimates for a circle. Of course, here you can calculate exactly the perimeter (aka circumference).

9. Making a comet. By taking \( a = -18, \ b = 0, \ r_1 = 25, \) and \( r_2 = 5, \) you can achieve the orbit of a comet. This merely shows that the path of a comet, just like the earth, follows an elliptical path.

10. Taking this further, with for example \( a = -245, \ r_1 = 250, \) and \( r_2 = 25 \) we can approximate a hyperbola. In the picture below, you can see and example.

Nearly a hyperbola

**Teaching strategies and teaching tips**
1. Parametric representation causes some confusion, particularly to students just beginning to grasp function relations. Be sure to distinguish the two notions.

2. Parametric equations are more general than functions.

3. Be sure to show that every function can be put in parametric form: \( y = f(x) \) transforms to
   \[
   \begin{align*}
   x &= t \\
   y &= f(t)
   \end{align*}
   \]

4. The domain of the function is not the domain of the parametric equations, EXCEPT in the situation just above, where the conversion is from a function form to parametric form.

**Mentoring tips**

1. Emphasize that parametric equations are very powerful, and particularly useful in describing space curves.