Content Background for Module 5

The following information is designed to address issues and topics related to conic sections and parametric equations not contained in typical high school textbooks or curriculum guides. Though some of the information on conic sections and parametric equations is easily understood by reading about them below, other important information on conic sections and parametric equations are better presented in video format.

We invite you to learn more about conic sections and parametric equations by viewing the video segments accompanying this module. The videos available for viewing cover the following concepts:

- **Conic Sections** – A review of the conic sections and their general equations, along with examples of how to graph each conic section.
- **Parametric Equations** – A demonstration of how parametric equations can model motion and instructions on how to use the TI-83 graphing calculator to graph parametric equations. This video also has examples of how to use the TI-83 graphing calculator to help solve problems on parametric equations.
- A video illustrating how the graph of velocity vs. position is quite elliptic in nature is included in this module. (See “The pendulum” under the Online Applet section.)
- A video illustrating the typical motion of a planet in our solar system is included in this module. (See “Parametric Curves” under the Online Applet section.)

**What are conic sections?**

Conic sections are the sections formed when a plane intersects two cones connected at their points. When this intersection takes place, four different curves are possible: the circle, the ellipse, the hyperbola, and the parabola. See the animations in this module to see how each conic section is formed.

**What are the reflective properties of each conic section?** In an ellipse, a particle originating at one focal point is reflected towards the other focal point. In a parabola, a particle traveling parallel to the axis of symmetry is reflected towards the focus. Animations illustrating these facts are in this module.

**What are parametric equations?**

Parametric equations are a way to symbolically represent a curve. At this point, your students know all of the parent functions and the properties that translate the parent functions around the coordinate plane. These parent functions can be symbolically represented as a function of $x$. In addition, your students know the general equations for the four conic sections and the properties that translate these curves around the coordinate plane. These conic sections can be represented as an equation containing two variables. Thus, your students are currently familiar with representing curves as a function of $x$ and representing curves with an equation that contains two variables.

Now is an appropriate time to show your students other ways to symbolically represent curves. Thus, it is appropriate to teach them about parametric equations. By definition, parametric equations are two equations $x = f(t)$ and $y = g(t)$ where $(f(t), g(t))$ is a point on the curve.

Notice the independent variable, commonly called the parameter, in these parametric equations is $t$. The parametric equations $x = \cos(t), y = \sin(t)$, for example, give the $x$- and $y$-coordinates of all points on the curve when a certain value of $t$ is known. If, for example, we let $t = \pi$, then the
The $x$-coordinate corresponding to this value of $t$ is $x(\pi) = \cos \pi = -1$ and the $y$-coordinate is $y(\pi) = \sin \pi = 0$. So we know that the point $(-1, 0)$ is on the curve of $x = \cos(t)$, $y = \sin(t)$. We can use this process and generate a number of points on the curve of $x = \cos(t)$, $y = \sin(t)$ by choosing various values for $t$ as shown in the following table.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t) = \cos(t)$</th>
<th>$y(t) = \sin(t)$</th>
<th>$(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>$\sqrt{2}/2 \approx 0.7071$</td>
<td>$\sqrt{2}/2 \approx 0.7071$</td>
<td>(0.7071, 0.7071)</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>0</td>
<td>1</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>$-\sqrt{2}/2 \approx 0.7071$</td>
<td>$\sqrt{2}/2 \approx 0.7071$</td>
<td>(-0.7071, 0.7071)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$-1$</td>
<td>0</td>
<td>(-1, 0)</td>
</tr>
<tr>
<td>$5\pi/4$</td>
<td>$-\sqrt{2}/2 \approx 0.7071$</td>
<td>$-\sqrt{2}/2 \approx 0.7071$</td>
<td>(-0.7071, -0.7071)</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>0</td>
<td>$-1$</td>
<td>(0, -1)</td>
</tr>
<tr>
<td>$7\pi/4$</td>
<td>$\sqrt{2}/2 \approx 0.7071$</td>
<td>$-\sqrt{2}/2 \approx 0.7071$</td>
<td>(0.7071, -0.7071)</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>1</td>
<td>0</td>
<td>(1, 0)</td>
</tr>
</tbody>
</table>

Now we can plot these points to help determine the curve generated by the parametric equations $x = \cos(t)$, $y = \sin(t)$.

The graph is obviously that of a circle with radius 1. When working with parametric equations, it’s important to plot the points in order of increasing values of $t$ so we can determine the “path” the parametric equations form. According to the table generated above, the first point to be plotted is $(1, 0)$ because it corresponds to the lowest value of $t$. The second point that should be plotted is $(0.7071, 0.7071)$ because it corresponds to the next lowest value of $t$.

It is sometimes helpful to have your students think of this as a “dot-to-dot”, and they need to “connect the dots” so the first “dot” represents the point generated by the lowest value of $t$, the
second “dot” represents the next lowest value of $t$, etc. Following this process, we should plot the points in the order shown in the following graph.

![Graph showing points plotted in order](image)

The arrows on the figure above show the direction in which the points were plotted, giving us the path formed by the parametric equations. It’s common to use parametric equations to describe the path taken by some object. Thus, if the path of an object was represented by the parametric equations $x = \cos(t)$, $y = \sin(t)$, we would say the object’s path is that of a circle with radius 1, traveling in a counterclockwise direction. (Note: if the parametric equations were $x = \sin(t)$, $y = \cos(t)$, the path would be clockwise.)

**Why do we use parametric equations?**

There are a number of reasons why we use parametric equations:

1. Parametric equations can represent the path of a moving object (see ball example below).

2. Parametric equations allow us to graphically compare two quantities based on a third factor (a.k.a. parameter). For example, suppose a football is thrown in the air and the parametric equations $x = 1.5t$, $y = 12\sin(t)$ ($0 \leq t \leq \pi$) represent the position of the ball after $t$ seconds have elapsed (with $x$ representing the horizontal position of the ball in yards and $y$ representing the vertical position of the ball in yards). If we construct a table of values, as shown below, we can find the position of the ball after $t$ seconds.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x(t)$</th>
<th>$y(t)$</th>
<th>$(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>1.1781</td>
<td>8.4853</td>
<td>(1.1781, 8.4853)</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>2.3562</td>
<td>12</td>
<td>(2.3562, 12)</td>
</tr>
<tr>
<td>$3\pi/4$</td>
<td>3.5343</td>
<td>8.4853</td>
<td>(3.5343, 8.4853)</td>
</tr>
<tr>
<td>$\pi$</td>
<td>4.7124</td>
<td>0</td>
<td>(4.7124, 0)</td>
</tr>
</tbody>
</table>

When we plot these points, we are plotting the horizontal distance of the ball vs. the vertical distance of the ball and can trace the motion of the ball, as shown in the following.
Now suppose that we were given the parametric equations and the graph of the football shown previously, but did not have the table of values. We can use these parametric equations to find out how long it took for the ball to reach its maximum height. From the graph, we can approximate the maximum height is 12 yards. Thus, we can set \( y = 12 \sin(t) \) equal to 12 and then solve for \( t \).

\[
12 = 12 \sin(t) \\
1 = \sin(t) \\
\frac{t}{2} = \frac{\pi}{2} \approx 1.57 \text{ seconds}
\]

This example clearly shows the beauty of parametric equations. We can find four important factors about this experiment (time elapsed, horizontal distance traveled, vertical distance traveled, path of the ball), yet only need a two-dimensional graph.

3. Most graphing calculators can graph parametric equations, which allow students to view graphs of curves that are not functions of \( x \). On the TI-83 graphing calculator, for example, when the student uses the Y= button, they can only type in equations of curves that are functions of \( x \). Changing their calculator to parametric mode allows students to view a larger variety of curves.
### What general parametric equations should my students know?

Pre-calculus students should be able to recognize the curves of the following parametric equations.

<table>
<thead>
<tr>
<th>Parametric Equations</th>
<th>Type of Curve</th>
<th>Describe the Path Formed by the Equations</th>
</tr>
</thead>
</table>
| \( x = a + b \cos(t) \)  
| \( y = c + b \sin(t) \) | Circle with radius \( b \) and center at \((a, c)\). | Travels in a counterclockwise direction. |
| \( x = a + b \sin(t) \)  
| \( y = c + b \cos(t) \) | Circle with radius \( b \) and center at \((a, c)\). | Travels in a clockwise direction. |
| \( x = a \sec(t) \)  
| \( y = b \tan(t) \) | Hyperbola (opening to the right/left) | Object would start on the positive \( x \)-intercept and travel upward, then wraps to the bottom portion of the left side of the hyperbola and travels upward, finally it wraps around on the bottom portion of the right hyperbola and finishes the path by traveling upward to the point where the object started. |
| \( x = a \tan(t) \)  
| \( y = b \sec(t) \) | Hyperbola (opening up/down) | Object would start on the positive \( y \)-intercept and travel to the right, then wrap to the left side of the bottom portion and travel to the right, finally it wraps to the left side of the top portion and finishes the path by traveling to the right to the point where the object started. |
| \( x = at^2 \)  
| \( y = 2at \) | Parabola (opening to the right) | Object starts on the bottom portion of the parabola and travels upward. (i.e. starts in the negative portion and travels toward the positive portion). |
| \( x = 2at \)  
| \( y = at^2 \) | Parabola (opening upward) | Object starts on the left portion of the parabola and travels toward the right. (i.e. starts in the negative portion and travels toward the positive portion). |
| \( x = a + b \cos(t) \)  
| \( y = c + d \sin(t) \) | Ellipse with center \((a, c)\), axis in \( x \) direction is of length \( 2b \), and axis in \( y \) direction is of length \( 2d \). | Travels in a counterclockwise direction. |
| \( x = a + b \sin(t) \)  
| \( y = c + d \cos(t) \) | Ellipse with center \((a, c)\), axis in \( x \) direction is of length \( 2b \), and axis in \( y \) direction is of length \( 2d \). | Travels in a clockwise direction. |
| \( x = a + bt \),  
| \( y = c + dt \) | Line containing the point \((a, c)\) with slope \( m = \frac{d}{b} \). | Travels from left to right. |

Following are some curves generated by the parametric equations given in the table above. Arrows represent the path formed by the parametric equations. It is important for you to investigate how the change in \( a, b, c, \) and \( d \) affects the curves. (Note: The starting point for some of these parametric curves depends on the values of \( t \) that are used.)
Circles

\[ x = 2 \sin(t), \quad y = 2 \cos(t) \]

\[ x = 3 \cos(t), \quad y = 3 \sin(t) \]

Horizontal Hyperbolas (these graphs are for \( t = -8\,..\,8 \))

\[ x = \sec(t), \quad y = \tan(t) \]

\[ x = 2 \sec(t), \quad y = 2 \tan(t) \]

\[ x = 4 \sec(t), \quad y = 4 \tan(t) \]

Vertical Hyperbolas (these graphs are for \( t = -8\,..\,8 \))

\[ x = \tan(t), \quad y = \sec(t) \]

\[ x = 3 \tan(t), \quad y = 3 \sec(t) \]

\[ x = 4 \tan(t), \quad y = 4 \sec(t) \]
\[ x = 2 \tan(t), \ y = 4 \sec(t) \]
\[ x = 3 \tan(t), \ y = \sec(t) \]
\[ x = \tan(t), \ y = 3 \sec(t) \]

Parabolas (these graphs are for \( t = -10..10 \))

\[ x = t^2, \ y = 2t \]
\[ x = 2t^2, \ y = 4t \]
\[ x = 3t^2, \ y = 6t \]

\[ x = 2t, \ y = t^2 \]
\[ x = 4t, \ y = 2t^2 \]
\[ x = 8t, \ y = 4t^2 \]
Ellipses

\[ x = 2 \cos(t), \ y = \sin(t) \]
\[ x = \cos(t), \ y = 3 \sin(t) \]
\[ x = 2 + 2 \cos(t), \ y = 1 + \sin(t) \]

\[ x = 2 \sin(t), \ y = 5 \cos(t) \]
\[ x = 2 + 4 \sin(t), \ y = 4 + 2 \cos(t) \]
\[ x = 2 \sin(t), \ y = -2 + 4 \cos(t) \]

Lines (these graphs are for \( t = -10..10 \))

\[ x = 2 + 4t, \ y = -2 + t \]
\[ x = 3t, \ y = -t \]
\[ x = 6t, \ y = 2 + 5t \]
What are the techniques my students can use to convert parametric equations to Cartesian equations (i.e. parametric form to rectangular form)?

- Solve both parametric equations for \( t \) and then set the two expressions for \( t \) equal to each other.
- Solve one equation for \( t \) and substitute it into the other equation.
- Use the trigonometric identities \( \sin^2(x) + \cos^2(x) = 1, \tan^2(x) + 1 = \sec^2(x) \).