Famous sequences

1. Constant sequence. Every term is the same. Need a base $a$. The terms are $a, a, a, a, \ldots$

2. Integers. Successive integers.

1, 2, 3, 4, 5, \ldots, $n$, \ldots

3. Square numbers/power numbers. Successive squares or successive powers.

1, 4, 9, 16, \ldots, $n^2$, \ldots  
Power is 2

1, 8, 27, 64, 125, \ldots, $n^3$, \ldots  
Power is 3

1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, \ldots, $\frac{1}{n^2}$, \ldots  
Power is $-2$

4. Even and odd numbers.

2, 4, 6, 8, 10, \ldots, $2n$, \ldots

1, 3, 5, 7, 9, 11, 13, \ldots, $2n + 1$, \ldots

5. Triangular numbers: Increasing by three each term.

1, 3, 6, 10, 15, \ldots, $\sum_{j=1}^{n} j$, \ldots

In the general term $n$ begins with $n = 0$.

6. Fibonacci: The next term is the sum of the previous two. The first two terms must be given. They are 1 and 1.

1, 1, 2, 3, 5, 8, 13, 21, \ldots

For your interest, there is a formula for the Fibonacci numbers:

$$f(n) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right)$$

Just substitute in successive integers $n$ and the successive terms are produced. Variations of the Fibonacci sequence can be obtained by beginning with two different generators. For example, let the generators be 1,3. Then the terms are

1, 3, 4, 7, 11, 18, 29, 47, \ldots

7. Geometric sequence: Need a base $a$. Then the terms are

$a, a^2, a^3, a^4, \ldots, a^n, \ldots$

**Example** $a = 2$. Terms:

2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384

**Example** $a = \frac{1}{2}$. Terms:

$\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$, $\frac{1}{512}$, $\frac{1}{1024}$, $\frac{1}{2048}$, $\frac{1}{4096}$, $\frac{1}{8192}$, $\frac{1}{16384}$
Example \( a = -\frac{2}{3} \). Terms:

\[
\begin{array}{cccccccccccc}
\frac{2}{3} & 4 & 9 & -8 & 27 & 16 & 81 & -32 & 243 & 64 & 729 & -128 & 2187 & 256 & 6561 & -512 & 19683 & 1024 & 59049 & -2048 & 177147 & 4096 & 531441 \\
\end{array}
\]


\[
1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{n}, \ldots
\]

This is a power sequence with power \(-1\).

9. Continued fraction. The one-over and over and over ... Terms:

\[
1, \frac{1}{1 + 1}, \frac{1}{1 + \frac{1}{1+1}}, \frac{1}{1 + \frac{1}{1 + \frac{1}{1+1 + \frac{1}{1+1 + \ldots}}}}
\]

Note how the general term is written, with the "dots." Here are the computed values.

\[
1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{8}{\sqrt{5} - 1}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \frac{55}{89}, \frac{89}{144}, \frac{144}{233}, \frac{233}{377}, \frac{377}{610}, \frac{610}{987}
\]

Do you see the Fibonacci sequence in there? How do we find the limit? Solve

\[
x = \frac{1}{1 + x}
\]

\[
x = \frac{1}{2} \sqrt{5} - \frac{1}{2}
\]

The equation is quadratic, but we take only the positive root. (Why?) We don’t have to use only the one’s above. The successive numerators can be any value. For example, consider

\[
2, \frac{2}{1 + 2}, \frac{2}{1 + \frac{2}{1+2}}, \frac{2}{1 + \frac{2}{1 + \frac{2}{1+2 + \frac{2}{1+2 + \ldots}}}}
\]

10. Primes. Numbers with no divisor other than themselves and one. No general formula is known.

\[
\]

The largest known prime is a Mercenne prime, a prime of the form \(2^n - 1\) - where \(n\) itself is prime. It has more than six million digits.

11. Recursions. Let \(f(x)\) be any function, and let \(a_1\) be given. Define \(a_{n+1} = f(a_n)\). Then the sequence \(a_1, a_2, a_3, \ldots\) is recursively generated. This sequence must have a starting value. Examples:

- a. Let \(f(x) = x^2\), and let \(a_1 = 2\). The first few terms of this sequence are

\[
2, 4, 16, 256, 65536, 4294967296
\]

Each is the square of the previous. These numbers approximately double the number of digits each step.

- b. Let \(f(x) = \frac{1}{1+x}\), and let \(a_1 = 1\). Then we can generate the terms above of the continued fraction.

\[
1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \frac{55}{89}, \frac{89}{144}, \frac{144}{233}, \frac{233}{377}, \frac{377}{610}, \frac{610}{987}
\]

Naturally, with a function of two variables, it is possible to generate some very
complex sequences. Among them is the “simple” Fibonacci sequence.

12. **Products.** The most important sequence formed by a product is the **factorial** sequence, 1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, 479001600, ... The general term is defined by

\[ n! = n(n - 1)(n - 2) \cdots (3)(2)(1) \]

with the special definition 0! = 1. In words, multiply the successive integers up to n to obtain \( n! \), read as ‘n factorial.’