Basic Transformations on Functions - what’s next?

The beauty of the parent functions is that they give a primitive form of the basic building blocks of functions. In this short “fast-forward” essay we offer a glimpse of what is to come mathematically for the student well grounded with an understanding of parent functions.

With the new added features such as shifts and translations we gain entire families of functions that are in many ways entirely similar to the original. Basically, we can wrap up all of the transformations in one equation. Given $y = f(x)$, consider the transformation

$$y = mf(ax + b) + c$$

This shifts and translates, magnifies, as well as dilates the function. In this manner a parent function can be “framed” in any desired interval, large or small. We now have an incredible class of functions to work with. But it doesn’t give all the functions we need.

Polynomials

We just need the operation of addition of functions. In this way we can generate representatives of every function used in engineering and physics. For example, we can consider polynomials

$$p(x) = a + bx + cx^2 + \cdots + zx^n$$

though it is often written as

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

In this way we can easily correspond the coefficient, by label, to the powers. Polynomials form an important class of functions for modeling and for approximating. The have been used for centuries for the widest variety of applications. Polynomials have interesting properties and look something like this one:

$$y = 0.9(x + 1)x^2(x - 1)(x - 1.5)(x - 2)$$

$$= 3.15x^3 - 2.7x^2 + 1.8x^4 - 3.15x^5 + 0.9x^6$$
They lead naturally to infinite polynomials (power series) such as

\[ P(x) = a_0 + a_1 x + a_2 x^2 + \cdots \]

\[ = \sum_{j=0}^{\infty} a_j x^j \]

This ancient class of functions was actually used before calculus was invented, but nowadays they are studies as a portion of college calculus. Without an understanding of parent functions, this important tool of mathematics, power series, are impossible to understand. In summary, we have the chain of understanding

Parent functions → Shifts, etc → Polynomials → Power series

Engineers, physicists, and scientists of all kinds use polynomials for a wide variety of reasons.

**Trigonometric functions**

There is a trigonometric analogue of polynomials, not surprisingly called trigonometric polynomials. Let’s consider the sine function, sin \( x \), which is of course one of the parent functions. Then we have

\[ m \sin(ax + b) + c \]

While this notation is a little non standard, it reveals that the periodic nature of the sinusoid can be placed anywhere along the \( x \)-axis raised up or down. These functions have the remarkable periodic property that so many phenomena obey. Even such applications as the price of gasoline has a periodic nature. In the illustration below we see temperature having a periodic nature.
Indeed many natural and business phenomena are said to have a ‘cyclic’ nature because of this. Economists and business executives need to understand periodicity well. Millions of dollars can change hands on the basis of just a slight about of extra information about price fluctuation. Now it is the case that not just one trig function will do. We need families of them. The general form is

\[ T_n(x) = a_0 + a_1 \sin x + a_2 \sin(2x) + \cdots + a_n \sin(nx) \]

This wonderful function, a trigonometric polynomial, is used to a variety of ends including such diverse applications as creating music files for the computer. In fact, one can view digitized music as collections of coefficients that the player assembles as above into a waveform that is amplified and played through the speakers.

The next level of trigonometric polynomials corresponds directly to power series. That is, consider the infinite trigonometric polynomial

\[ T(x) = \sum_{j=1}^{\infty} a_j \sin(jx) \]

and its more general cousin

\[ F(x) = b_0 + \sum_{j=1}^{\infty} a_j \sin(jx) + b_j \cos(jx) \]

More frequently called a Fourier series, they are named after the French mathematician Joseph Fourier (1768-1830) and military officer. He first applied such series to the study of heat flow, to the disdain of many other mathematicians. Even in his lifetime the premises Fourier made in his studies were repudiated by some. However, today Fourier series form a central part of applied and pure mathematics. Fourier series are encountered in the second year of college. Here is the chain

Parent functions → Shifts, etc → Trigonometric Polynomials → Fourier series
Notice the similarity with the polynomial to power series chain. This idea applies to much of function theory and mathematics in general. Begin with a basis of information, then expand using simple operations. Next, combine using additive or possibly multiplicative operations. Finally, extend to infinity.

**Its in the air**

As well trigonometric functions and polynomials are use to encode AM and FM radio signals. There is a nifty applet in this sections that shows how various types of radio signals encode the sound, sometimes as amplitude modulation (AM) and sometimes as frequency modulation (FM). In these cases we consider a produce of sinusoids. For example, the double side band suppressed carrier wave (AM) has the form

\[ c \sin(at) \cos(bt) \]

The picture below has that shape: \( 5 \sin(40t) \sin(2.2t) \). In fact, this is a trigonometric polynomial as you can derive using trigonometric identities.

5 \( \sin(40t) \sin(2.2t) \)

The applications of trigonometric polynomials pervades almost every endeavor where periodic behavior is evident. Only in the past few years have trigonometric polynomials been replaced by a new and more successful tool, that of wavelets.

**Summary**

We can see clearly that the parent functions form the basis of what is to come mathematically. They form the building blocks of the complex models used to describe many of today’s phenomena. Even the models used in creating movies such as “Lord of the Rings” required software engineers to have a solid understanding of these functions. It is important for students to buy-in to the fact that the parent functions form the starting point for the study of very useful mathematics, in that these functions enable them to describe any mathematical
function, shape or object they may need.

References

Text


Stewart, James, Calculus (Early Transcendentals), 3rd Ed., Brooks/Cole, Pacific Grove, CA, 1995

On the Web

