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Medieval Mathematics

Because much mathematics and astronomy available in the 12th century was written in Arabic, the Europeans learned Arabic. By the end of the 12th century the best mathematics was done in Christian Italy. During this century there was a spate of translations of Arabic works to Latin. Later there were other translations.

\[
\text{Arabic} \rightarrow \text{Spanish} \\
\text{Arabic} \rightarrow \text{Hebrew} \quad (\rightarrow \text{Latin}) \\
\text{Greek} \rightarrow \text{Latin}.
\]

Example. *Elements* in Arabic $\rightarrow$ Latin in 1142 by Adelard of Bath (ca. 1075-1160). He also translated Al-Khwarizmi’s astronomical tables (Arabic $\rightarrow$ Latin) in 1126 and in 1155 translated Ptolemy’s Almagest (Greek $\rightarrow$ Latin) (The world background at this time was the crusades.)

1 Gherard of Cremona (1114 - 1187)

Gherard’s name is sometimes written as Gerard. He went to Toledo, Spain to learn Arabic so he could read Ptolemy’s Almagest since no Latin translations existed at that time. He remained there for the rest of his life. Gherard made translations of Ptolemy (1175) and of Euclid from Arabic. Some of these translations from Arabic became more popular than the (often earlier) translations from Greek. In making translations of other Arabic work he translated the Arabic word for sine into the Latin *sinus*, from where our *sine* function comes. He also translated Al-Khwarizmi.

2 Adelard of Bath, (1075 - 1160)

During this period (12th century) the Hindu numerals became known to Latin readers by Adelard of Bath, also known as Robert of Chester. Adelard studied and taught in France and traveled in Italy, Syria and

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Palestine before returning to Bath. He was a teacher of the future King Henry II. Adelard translated Euclid’s *Elements* from Arabic sources. The translation became the chief geometry textbook in the West for centuries. He translated al’Khwarizmi’s tables and also wrote on the abacus and on the astrolabe. One book, his *Quaestiones naturales* consists of 76 scientific discussions based on Arabic science.

3 Leonardo Pisano Fibonacci (1170 - 1250)

Fibonacci or Leonard of Pisa, played an important role in reviving ancient mathematics making significant contributions of his own. Leonardo Pisano is better known by his nickname Fibonacci. He played an important role in reviving ancient mathematics and made significant contributions of his own. Fibonacci was born in Italy but was educated in North Africa where his father held a diplomatic post. He traveled widely with his father, recognizing and the enormous advantages of the mathematical systems used in these countries.

Fibonacci *Liber abaci (Book of the Abacus)*, published in 1202 after his return to Italy, is based on bits of arithmetic and algebra that Fibonacci had accumulated during his travels. Liber abaci introduced the Hindu-Arabic place-valued decimal system and the use of Arabic numerals into Europe. *Liber abaci* did not appear in print until the 19th century. A problem in *Liber abaci* led to the introduction of the Fibonacci numbers and the Fibonacci sequence for which Fibonacci is best remembered today. The *Fibonacci Quarterly* is a modern journal devoted to studying mathematics related to this sequence.

Fibonacci’s other books of major importance are *Practica geometriae* in 1220 containing a large collection of geometry and trigonometry. Also in *Liber quadratorum* in 1225 he approximates a root of a cubic obtaining an answer which in decimal notation is correct to 9 places.
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Features of \textit{Liber abaci}: 

- a treatise on algebraic methods and problem which advocated the use of Hindu-Arabic numerals.
- used the horizontal bar for fractions.
- in fractions though the older systems of unit and sexagesimal were maintained!
- contained a discussion of the now-called \textit{Fibonacci Sequence} – inspired by the following problem:

“How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on.”

The sequence is given by

\[ 1, 1, 2, 3, 5, 8, 13, 21, \ldots, u_n, \ldots \]

which obeys the recursion relation

\[ u_n = u_{n-1} + u_{n-2} \]

- Some of Fibonacci’s results:

\textbf{Theorem.} (i) Every two successive terms are relatively prime.

(ii) \( \lim_{n \to \infty} \frac{u_{n-1}}{u_n} = (\sqrt{5} - 1)/2. \)

\textbf{Proof.} (i) \( u_n = u_{n-1} + u_{n-2}. \) If \( p|u_n \) and \( p|u_{n-1} \), then \( p|u_{n-2} \Rightarrow p|u_{n-3} \ldots \Rightarrow p|u_1. \) #.

(ii) From \( u_n = u_{n-1} + u_{n-2} \) we have

\[ 1 = \frac{u_{n-1}}{u_n} + \frac{u_{n-2}}{u_n} = \frac{u_{n-1}}{u_n} + \frac{u_{n-2}}{u_{n-1}} \cdot \frac{u_{n-1}}{u_n}. \]

So, if \( \lim \frac{u_{n-1}}{u_n} \) exists and equals \( r \), it follows that

\[ 1 = r + r^2 \quad r = \frac{-1 \pm \sqrt{5}}{2} = \frac{\sqrt{5} - 1}{2}. \]
This is a *golden section connection*. To show that $u_{n-1}/u_n$ converges, define $s_n = u_{n-1}/u_n$. Clearly, $u_{n-1}/u_n \geq \frac{1}{2}$. Then

$$1 = s_n(1 + s_{n-1}) \quad \text{or} \quad s_n = \frac{1}{1 + s_{n-1}}$$

$$f(x) = \frac{1}{1+x}, \quad f'(x) = -\frac{1}{(1+x)^2}. \quad \text{If } x > 0 \quad |f'| < 1$$

$$|s_n - s_{n-1}| = |f(s_{n-1}) - f(s_{n-2})| < f'(\alpha_n)|s_{n-1} - s_{n-2}| < \beta_n|s_{n-1} - s_{n-2}|.$$  

Since $\beta_n < k < 1$, this establishes convergence.

Alternatively: $s_{n+2} = \frac{1}{1 + s_{n+1}}$. This is a decreasing sequence because $s_0 > \frac{\sqrt{5} - 1}{2}$. So $s_0 > s_2 > s_4 > \cdots$. Now give a lower bound using $s_1, s_2, \ldots > \ldots$. Next show that the limits must be the same. etc,etc,etc.

Other properties:

$$u_1 = u_3 - u_2$$
$$u_2 = u_4 - u_3$$
$$u_3 = u_5 - u_4$$
$$\vdots$$
$$u_{n-1} = u_{n+1} - u_n$$
$$u_n = u_{n+2} - u_{n+1}.$$  

So to get

$$\sum_{j=1}^{n} u_j = u_{n+2} - u_2.$$  

This formula can also be used to prove that $\lim u_{n-1}/u_n$ exists. Also

$$u_{n+1}^2 = u_n u_{n+2} + (-1)^n \quad \text{(prove by induction)}.$$
The Pascal triangle connection.

Beginning with each one (1) going down the left diagonal, sum up the diagonal entries where the diagonal slope is 1/3 (i.e. 3 cells right, 1 cell up, ...). This scheme generates the Fibonacci sequence.

The modern, general form: Given \( a, b, c \), and \( d \). Let

\[
x_0 = a \quad x_1 = b \quad x_{n+2} = cx_{n+1} + dx_n.
\]

There are many results about such sequences, some similar to those already shown.

**A cubic equation.** In what appears at an attempt toward proving that solutions of cubic equations may not be constructable numbers, Fibonacci showed that the solution to the cubic equation

\[
x^3 + 2x^2 + 10x = 20
\]

can have no solution of the form \( a + \sqrt{b} \), where \( a \) and \( b \) are rational. He gives an approximation 1; 22, 7, 42, 33, 4, 40 – best to that time, and for another 300 years. Note the use of sexagesimal numbers.

He also wrote *Liber quadratorum*, a brilliant work on intermediate analysis. Consider the Diophantine-like problem posed by **Master John of Palermo**. The number 5, added or subtracted from the square, the result will be the square of a rational. In modern form

\[
r^2 + 5 = s^2 \quad r^2 - 5 = t^2 \quad r, s, t \text{ rational}
\]
The solution of this problem appears as Proposition 17 of the 24 propositions in the work. Fibonacci’s resolution is remarkably sophisticated. First, he defines the notion of **congruous numbers**: numbers of the form \( ab(a+b)(a-b) \) if \( a+b \) is even or \( 4ab(a+b)(a-b) \) if \( a+b \) is odd. Such numbers he shows must be divisible by 24. Moreover, the system \( x^2 + m = s^2 \) and \( x^2 - m = t^2 \) has integers solutions only if \( m \) is congruous. Next, he shows that 5 is not congruous, but \( 12^2 \cdot 5 \) is congruous. From this he is able to find a rational solution. Answer: \( \frac{3\sqrt{5}}{12} \).

In *Liber quadratorum* Fibonacci makes frequent use of the identities
\[
(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (bc - ad)^2 = (ad + bc)^2 + (ac - bd)^2,
\]
which had also appeared in Diophantus.

**Liber abaci** summary:

- **Sources** – Islamic texts
- **Contents**
  - Rules for positional arithmetic,
  - Rules for the calculation of profits, currency, conversions, measurement
- **Problem types** – mixture problems, motion problems, container problems, Chinese remainder problem, quadratics, summing series
- **Methods** – wide and varied – most are original

**Another Examples.** (A) If you give me a coin, we have the same.

(B) If I give you a coin, you have ten times what I have.

\[
x = \text{me} \\
y = \text{you} \\
z = x + y
\]
is the total amount.

**Solution.** Add \( x + 1 \) to both sides of the first equation to get
\[
x + 1 = \frac{1}{2}z \\
y + 1 = \frac{10}{11}z.
\]
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So \( x + y + 2 = \left( \frac{1}{2} + \frac{10}{11} \right) z = \frac{31}{22} z \)

\[
\begin{align*}
\frac{9}{22} z &= 2 \\
z &= \frac{44}{9} \\
x &= \frac{44}{18} - 1 = \frac{26}{18} = \frac{13}{9} \quad y = \frac{32}{9}
\end{align*}
\]

4 Medieval Universities

The modern university evolved from medieval schools known as *studia generalia*, recognized places of study open to students from all of Europe. As indicated earlier, these *studia* were created from the need to educate clerks and monks, at a level beyond the monastic schools. They included scholars from other countries, and this constituted a primary difference between the *studia* and the schools from which they grew.

The very earliest Western institution that could be termed a university was a 9th century medical school at Salerno, Italy. Drawing students from all of Europe, it was reknown as a medical school. The first true universities, comprised of many disciplines were founded at Bologna late in the 11th century, the University of Paris, founded between 1150 and 1170, and the University of Oxford in England, which was well established by the end of the 12th century. The later two were composed of colleges, which in fact were endowed residence halls for scholars.

These universities were societies/guilds enjoyed wide ranging indepenence, given at the discretion of kings, emperors, and popes, who only required that neither heresy or atheism could be taught. The price of the independence was that they pay their own way. This required that the scholars charged tuition to gain a livlihood. As such they needed to satisfy the students on whom they depended for fees. As a consequence various universities were in vogue or not as hosts of students might migrate from one institution to another. Indeed, the University of Cambridge was founded by disgruntled students from Oxford.

From the 13th century onwards, all major cities had a university.
The curriculum consisted of the classical *trivium* and *quadrivium* of the classical age of Greece. Thus, the subjects studied were logic - grammar - rhetoric - arithmetic - geometry - music - astronomy. Study focused on the works of the great philosophers such as Aristotle and Plato. Mathematical studies included the texts by Euclid and Nicomachus.

The first university, modern secular sense, was founded in Halle\(^2\) (Germany) by Lutherans in 1694. This school renounced religious orthodoxy of all kinds, favoring rational and objective intellectual inquiry. Also, lectures were given in the vernacular (German) instead of Latin. Halle’s innovations were later adopted by the University of Göttingen (founded in 1737) and subsequently by most universities.

5 Jordanus Nemorarius

(fl. 1220) Jordanus (fl. 1220) was younger than Fibonacci and was the founder of the Medieval school of **mechanics**. Almost nothing about him is known, except it is believed he taught at the new university in Paris about 1220. He wrote on geometry, arithmetic and mechanics. His fame was assured by his solution of a problem that eluded Archimedes, namely the problem of the inclined plane,

\[
F = mg \sin \theta.
\]

He wrote the book *Arithmetica*. In it he demonstrates a mastery of the theory of proportion as well as facility with quadratic terms. Some examples of his results:

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\(^{2}\)The great set theorist, Georg Cantor, was a faculty member at Halle.
Theorem. A multiple of a perfect\textsuperscript{3} number is abundant. A divisor of a perfect number is deficient.

Note the rhetorical base significant for the use of letters instead of numerals for numbers. The uniqueness of the solution of division problems is considered by Jordanus as follows:

Theorem. If a given number is divided in two parts whose difference is given, each of the parts is determined.

Theorem. If a given number is divided into two parts whose product is determined each of the parts is determined.

Theorem. If a given number is divided into however many parts, whose continued proportions are given, then each of the parts is determined.

6 Nicole Oresme

Nicole Oresme (1323 - 1382), after studying theology in Paris, became bursar in the University of Paris and later dean of Rouen. In 1370 he was appointed chaplain to King Charles V as his financial advisor.

Oresme invented coordinate geometry before Descartes whereby he established the logical equivalence between tabulated values and their graphs. He proposed the use of a graph for plotting a variable magnitude whose value depends on another. It is possible that Descartes was influenced by Oresme’s work since it was reprinted several times over 100 years after its first publication. Oresme also worked on infinite series which we shall discuss presently.

Another work by Oresme contains the first use of a fractional exponent, although, of course, not in modern notation. Oresme also opposed the theory of a stationary Earth as proposed by Aristotle and taught motion of the Earth — 200 years before Copernicus.

Latitude of forms. in about 1361 he conceived of the idea to visualize or picture the way things vary (function representation at an early stage). Everything measurable, Oresme wrote, is imaginable in the manner of continuous quantity. In this way he drew a velocity-time graph for a

\textsuperscript{3}Recall that a number is termed perfect if its divisors add up to itself. A number is abundant if its divisors sum to a number greater than itself. A number is deficient if its divisors sum to a number smaller than itself. For example, 6 and 28 are perfect; 12 and 36 are abundant; 7 and 21 are deficient. In general, all prime numbers are deficient.
body moving with uniform acceleration. In this connection he used the terms latitude and longitude as we use abscissa and ordinate. His graphical representation is akin to our analytic geometry. His use of coordinates was not new however. (Apollonius) His main interest was in quadratures, and therefore he missed noticing functions and functional ideas *per se*.

The graphical representation of function, known as the *latitude of forms* was a popular topic from the time of Oresme to Galileo. His *Tractatus de figuratione potentiarum et mensurarum* was printed four times between 1482 and 1515. Oresme even suggested a three dimensional version of his latitude of forms.

Oresme generalized Bradwardine’s rule of proportion to include fractional powers giving the equivalents of our laws of exponents

\[ x^m x^n = x^{m+n} \quad (x^m)^n = x^{mn} \]

suggested the use of irrational powers \( x^{\sqrt{2}} \) but failed due to terminology and notation and . . .

6.1 The Merton school

The Merton School was one of the colleges at Oxford where a step toward a more modern physics was advanced by Thomas Bradwardine. One of his main interests was the investigation of infinite decomposibility of the continuum. He also considered geometrical shapes in terms of the points that comprise them. One problem that he exposed dated from Euclid’s time, that being the angle between a curve and its tangent. He argued that if the angle is positive there results a contradiction, while if it is zero there can be no angle. A paradox? One of his greatest efforts was toward proving the **Mean Speed Rule** – distance traveled by an object in uniform acceleration, namely that the mean speed achieved halfway through the accelerated motion. Graphically, we have
He goes further. Consider this:

**Halves** Area 1st half : Area 2nd half = 1 : 3 (Mean Speed Rule)

**Thirds:** \( A_{1st} : A_{2nd} : A_{3rd} = 1 : 3 : 5 \)

**Fourths:** \( A_{1st} : A_{2nd} : A_{3rd} : A_{4th} = 1 : 3 : 5 : 7 \)

and so on. In as much as the sums of the odd integers are \( n^2 \), the total distance covered varies as the square of time * Galileo: law of motion.

### 7 Infinite series

In the fourteenth century mathematicians had imagination and precision of thought, but lacked algebraic and geometric facility. Hence, they could at most equal the ancients in the same area. However, as we have seenm they ventured into new areas. Another direction was toward infinite series. We emphasize here again the importance of a new philosophical viewpoint which permits new thoughts unconfined by past taboos. Nicole Oresme ventured boldly in this new direction without the *horror infiniti* of the Greeks. Among the many series he summed was:

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots = 2
\]

He produced a beautiful geometric proof.
This appearance of computing infinite sums may be illusory if taken from Oresme’s viewpoint. Recall that at the time, philosophy was still very much Aristotelian, and Aristotle’s conception of infinity was essentially temporal. An infinite process is one which does not end. But could this be applied to permanent objects? There were tensions.

Aristotle views that continuous objects were infinitely divisible. But when one divides a length into halves, then one of the halves into halves, one of those quarters into halves and so on indefinitely. Are those pieces really there? Aristotle would say they are only potentially there.

Mathematicians such as William of Ockham and Gregory of Rimini maintained that they were indeed there — but there was no last member. This is the usual problem in dealing with the infinite.

An interpretation. What Oresme may have been doing is experimenting with moving these parts around. He conceived of two squares, each a foot in length. He divided the second square into proportional parts by means of vertical slices. These parts were then moved one on top of the other, creating a vertical tower. The total area was two square feet. Oresme’s main interest, however, was in showing how a finite object could be infinite in this respect.
Robert Suiseth, (or Swineshead) (fl. ca. 1350), an English logician, better known as Calculator solved the following infinite series problem:

If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double this intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval.

This is equivalent to saying the sum of the series

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots = 2.
\]

Calculator gave a long and tedious proof, not knowing the graphical representation.

Alternate proof: (modern)

\[
\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots
\]

\[
= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

\[
+ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

\[
+ \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\]

\[
+ \cdots
\]

\[
= 1 + \frac{1}{2}(1) + \frac{1}{4}(1) + \cdots
\]

\[
= 2
\]

* Oresme also summed

\[
\frac{1 \cdot 3}{4} + \frac{2 \cdot 3}{16} + \frac{2 \cdot 3}{64} + \cdots + \frac{2 \cdot 3}{4^n} + \cdots = 2.
\]

* He proved the harmonic series diverges by grouping

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{n} + \cdots
\]
8 Decline of Medieval learning

- Black Plague
- Hundred Years War (England-France)
- War of the Ros
- Shift from England and France to
  - Italy
  - Poland
  - Germany