1 Exercises

1. Consider the arithmetic progression \(0, b, 2b, 3b, \ldots\). Suppose \((d, b) = 1\).\(^2\) Prove that the series \(\{kb \pmod{d}\}, \ k = 1, 2, \ldots\), contains \(d\) different residues. (Hint. Prove that the series \(\{kb \pmod{d}\}, \ k = 1, 2, \ldots\), contains \(d\) different residues.

2. For any integers \(a, b,\) and \(m\) show that \(ab \pmod{m} = \[a \pmod{m} b \pmod{m}\] \pmod{m}\).

3. Let \(s = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \cdots\) be the sum of the reciprocals of all numbers with primes factors 2, 3, and 5. Prove Euler’s formula in the special case, that \(\prod_{p=2,3,5} \frac{1}{1-p^{-1}}\).

4. Compute \(\varphi(25), \varphi(32),\) and \(\varphi(100)\).

5. Show that \(\varphi(2n) = \varphi(n)\), for every odd integer \(n\).

6. Prove that if the integer \(n\) has \(r\) distinct primes, the \(2^2|\varphi(n)\).

7. Prove that the Euler \(\varphi\)-function is multiplicative. That is, \(\varphi(mn) = \varphi(m)\varphi(n)\). (This may prove difficult.)

8. Show that there is no odd perfect number which is the product of just two odd numbers \((\neq 1)\).

9. Prove the formula \(\ln(1-x^2) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)\)

10. Express \(\sqrt{i}\) in the form \(a + ib\).

11. Classify which numbers of the form \(\sqrt{p \sqrt{q}}\) are transcendental.

12. Use the classical result \(e^{ix} = -1\) and Gelfond’s theorem to establish that \(e\) cannot be algebraic.

13. Note two example of aspects of number theory that required further algebraic development ot solve.

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\(^2\)This means \(d\) and \(b\) are relatively prime.
Algebra and Number Theory

14. Explain the development of algebra as a consequence of symbolism. (Hint. What aspects of 19th century developments would have been impossible without symbolism?)

15. Write a short essay on the impact of number theory on the development of algebra.