Chapter 10. Nonlinear Systems

In this chapter we begin our study of nonlinear systems of differential equations. It is almost never possible to find closed form solutions to these systems. What we will substitute is

1. The determination of numerical solutions, and
2. A qualitative analysis of the solution in the vicinity of an equilibrium solution.

The fact of the matter is that very often we can approximate the nonlinear system by a linear system that has similar asymptotic behavior. While the analysis of planar linear systems is interesting, it is not very useful in and of itself. Its real power is revealed through the linearization process described below. Moreover, there are a great many interesting and useful nonlinear systems in two variables. So, our previous work will be put to good use.

Introduction

The basic system of this chapter is

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

We will always assume that \((x_0, y_0)\) is an equilibrium solution. This means, of course, that

\[ f(x_0, y_0) = 0 \quad \text{and} \quad g(x_0, y_0) = 0 \]

Through the predator-prey system and the SIR system, we have seen a variety of behaviors for solutions of nonlinear systems, sometimes with solutions converging to an equilibrium, sometimes orbiting around it, and sometimes diverging from it. At a point of reference, for the linear system \(x' = Ax\) previously discussed, the equilibrium solution is \(x_0 = 0\)

One dimensional problems

Let’s consider just the on dimensional problem

\[ y' = f(y) \]

with equilibrium solution \(y_0\). Then we can approximate \(f\) in a Taylor series about \(y_0\),

\[ f(y) \approx f(y_0) + f'(y_0)(y - y_0) \]
Recall, this is an equality if we add the error term
\[ f(y) = f(y_0) + f'(y_0)(y - y_0) + E_2(y - y_0)^2 \]
where \( E_2 = \frac{1}{2} f''(\bar{y}) \) and \( \bar{y} \) is between \( y \) and \( y_0 \). Our purpose here is not to explore Taylor approximations to the differential equation. Rather, we will restrict ourselves to linear approximations. The approximation to \( y' = f(y) \) is then the linear differential equation
\[ y' = f'(y_0)(y - y_0), \]
and this we transform to
\[ u' = f'(y_0)u \]
where \( u = y - y_0 \). You see we are now working in relative coordinate with respect to the equilibrium. For this relative coordinate system the equilibrium has been transformed to 0.

**Convergence to the equilibrium**

While we have discusses only particular cases, we have previously observed that if \( f'(y_0) < 0 \), the solution to the original differential equation converges to the equilibrium solution, provided the initial point is sufficiently close. (Refer to the SIR model for the details of the argument.) What is important here is that the linearized equation \( u' = f'(y_0)u \) has the solution \( u = u_0e^{f'(y_0)t} \) and with \( f'(y_0) < 0 \) it follows that
\[
\lim_{t \to \infty} u(t) = 0
\]
What is remarkable is that the inference goes the opposite way. If we can show the solutions converges or diverges (to 0) for the linearized problem, this implies the same for solutions to the original problem.

Conversely, if \( f'(y_0) > 0 \), a similar analysis showed that solution diverged. This is reflected in the linear system, as well.

**The NO information case**

The only situation where no information can be obtained is when \( f'(y_0) = 0 \). For example, consider the two differential equations
\[ y' = -y^2 \quad \text{and} \quad x' = x^2 \]
Here both have the equilibrium solution 0. It is also clear that the derivative \( f'(0) = 0 \) for both. BUT, the solution of the first is \( y(t) = \frac{1}{t + C_1} \) and the solution of the second is \( x(t) = \frac{1}{-t + C_1} \). However, for positive initial conditions (i.e. \( y(0) = x(0) = a > 0 \), etc.), the first converges while the second diverges!

Our intent is now to study similar phenomena for two dimensions. We will be able to conclude converge or divergence. There will also be a no information case — though it is far more subtle.

**Exercise**  *What is the asymptotic nature of these solutions when negative initial
Two dimensional problems

Let’s get back to the problem in question.

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

We will always assume that \((x_0, y_0)\) is an equilibrium solution, which means that \(f(x_0, y_0) = 0\) and \(g(x_0, y_0) = 0\). The linearization process uses the Taylor series in two dimensions this time. We note that

\[
\begin{align*}
  f(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E_{2,f} \\
  g(x, y) &= g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) + E_{2,g}
\end{align*}
\]

where \(E_{2,f}\) and \(E_{2,g}\) are the errors of the approximation. Neglecting these error terms we approximate the differential equation by the linear differential system

\[
\begin{align*}
  x' &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
  y' &= g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)
\end{align*}
\]

The matrix

\[
J(x_0, y_0) = \begin{bmatrix}
  f_x(x_0, y_0) & f_y(x_0, y_0) \\
  g_x(x_0, y_0) & g_y(x_0, y_0)
\end{bmatrix}
\]

is called the Jacobian matrix of \((f, g)^T\).

The linearized system.

The core of our analysis is here. Using the new variables \(u = x - x_0\) and \(v = y - y_0\) we write our linearized system in the relative coordinates - just as above

\[
\begin{align*}
  u' &= f_x(x_0, y_0)u + f_y(x_0, y_0)v \\
  v' &= g_x(x_0, y_0)u + g_y(x_0, y_0)v
\end{align*}
\]

or in matrix form

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}' = J
\begin{bmatrix}
  u \\
  v
\end{bmatrix}
\]
So, you can see that the dynamics of this system depends on the eigenvalues of the Jacobian $J$. Recall the possible events:

1. **Nodal sinks** (real negative eigenvalues)
2. **Nodal sources** (real positive eigenvalues)
3. **Spiral sinks** (negative real parts of complex eigenvalues)
4. **Spiral sources** (positive real parts of complex eigenvalues)
5. **Saddle points** (real positive and negative eigenvalues)
6. **Ellipses or centers** (zero real parts of eigenvalues)

The sixth event, elliptical orbits, occurred when the eigenvalues were purely complex, that is having zero real parts. This case will play the “no information” role analogous to the one dimensional case. The fundamental result about our situation is contained in the following

**Theorem** If the linearized system has dynamics specified by cases (1) - (5) above, so also will the original nonlinear system.

We will call the equilibrium **generic** if one of the cases (1)-(5) occur for the linearized system. So, we can re-phrase the above result by saying: If the linearized system has a generic equilibrium, so also does the original nonlinear system.

It’s time to consider some examples.

**Examples**

Review the examples in the text. Below we create a few examples with multiple equilibria, of differing types.

**Example** Analyze the system

\[
\begin{align*}
x' &= (.5 - \sin x)(-1 + \sin y) \\
y' &= (1 - \sin x)(.5 - \sin y)
\end{align*}
\]

**Solution** So, \(f(x,y) = (.5 - \sin x)(-1 + \sin y)\) and \(g(x,y) = (1 - \sin x)(.5 - \sin y)\). For this problem, there are many equilibrium points. For example \(\left(\frac{\pi}{6}, \frac{\pi}{6}\right)\) is an equilibrium. Also \((\pi, -\pi)\) is an equilibrium. In addition, we can see that any combination of \((x,y)\) from the values below yield an equilibrium.

\[
\begin{align*}
x &= \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, \ldots \pm(2k - 1)\frac{\pi}{2} \\
y &= \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}, \ldots, \pm(2k - 1)\frac{\pi}{2}
\end{align*}
\]

Thus for another example, the point \(\left(\pi - \frac{\pi}{6}, \frac{\pi}{6}\right) \approx (2.61, 0.52)\) is an equilibrium solution. In the vector field plot below, it is easy to see the equilibrium points in the limited region shown. The point designated above
appears to be a stable equilibrium, while the point \( \left( \frac{\pi}{6}, \pi - \frac{\pi}{6} \right) \) seems to be unstable.

The numerical solution of this system for a variety of starting values is shown below. All of the solutions appear to converge to the equilibrium solution \( \left( \pi - \frac{\pi}{6}, \frac{\pi}{6} \right) \).

Now let’s perform the linearization of this example and check to see if \( \left( \pi - \frac{\pi}{6}, \frac{\pi}{6} \right) \) is indeed a nodal sink, as it appears to be. Letting \( x_0 = \pi - \frac{\pi}{6} \) and \( y_0 = \frac{\pi}{6} \), the Jacobian matrix is
As is apparent, because the Jacobian is triangular, both eigenvalues are negative. Although this does not exactly fit previously considered cases (why?), we can conclude that \((\pi - \frac{\pi}{6}, \frac{\pi}{6})\) is a nodal sink. Therefore, by the fundamental theorem it is also a nodal sink for the original nonlinear system. For reference, had we selected the equilibrium solution \((\pi - \frac{\pi}{6}, \frac{\pi}{6})\)

Thus \((\pi - \frac{\pi}{6}, \frac{\pi}{6})\) is a nodal source. It is important to notice that even though it is impossible to solve the original differential equation, we know a great deal about the dynamics of the solution. Therefore, computing a numerical solution is perfectly fine. Finally, suppose we select the equilibrium \((\frac{\pi}{2}, \frac{\pi}{2})\). Then the Jacobian is

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which has the double eigenvalue \(\lambda = 0\). This is a center. No information can be obtained.

**Exercise** Exactly what kind of equilibrium does this problem have at \((\frac{\pi}{2}, \frac{\pi}{2})\)?

**Example** Analyze the equilibrium behavior of \(x' = y\) and \(y' = -9\sin x - \frac{1}{3}y\).
Solution This is an excellent example, showing alternating spiral sinks and saddle point. For the record, $f(x,y) = y$ and $g(x,y) = -9\sin x - \frac{1}{5}y$. First of all, it is easy to see that $(k\pi,0)$, $k = 0,\pm 1,\pm 2,\ldots$ are equilibria of this system. To linearize this system, we determine the Jacobian matrix

$$J(x,y) = \begin{bmatrix}
\frac{\partial}{\partial x} f(x,y) & \frac{\partial}{\partial y} f(x,y) \\
\frac{\partial}{\partial x} g(x,y) & \frac{\partial}{\partial y} g(x,y)
\end{bmatrix}$$

= \begin{bmatrix}
0 & 1 \\
-9 \cos x & -\frac{1}{5}
\end{bmatrix}

It has eigenvalues given by $\lambda = -\frac{1}{10} \pm \frac{1}{10} \sqrt{1 - 900 \cos x}$. Note that we have computed the eigenvalues for any $(x,y)$, not just at the equilibria. When we select for $x_0$ the odd multiples of $\pi$, namely $x_0 = \pm (2k-1)\pi$, we see that the eigenvalues are

$$\lambda = -\frac{1}{10} \pm \frac{1}{10} \sqrt{1 - 900 \cos((2k-1)\pi)}$$

= $-\frac{1}{10} \pm \frac{1}{10} \sqrt{1 + 900}$

= $2.902, -3.102$

This means the equilibria $(\pm (2k-1)\pi,0)$ are saddle points. On the other hand for even multiples of $\pi$, we have that

$$\lambda = -\frac{1}{10} \pm \frac{1}{10} \sqrt{1 - 900 \cos(2\pi)}$$

= $-\frac{1}{10} \pm \frac{1}{10} i \sqrt{899}$

The eigenvalues are complex with negative real parts. Therefore, these equilibria are spiral sinks. Below we show some typical solution curves in the phase plane.
As you can see, there have been numerous initial conditions including several near the saddle points \((\pm \pi, 0)\). However the convergence is always toward the stable equilibria, \((\pm 2\pi, 0)\) and \((0, 0)\) and away from the saddle points.

**Long term behavior of solutions**

Although we have already considered asymptotic behavior, we will do so a little more thoroughly in this section.