Tips for Teachers - on Quadratics

The Quadratic Formula.

Given \( ax^2 + bx + c \). In the case when \( a \neq 0 \), the solutions to \( ax^2 + bx + c = 0 \) are

\[
-\frac{b \pm \sqrt{-4ac + b^2}}{2a}
\]

In the case \( a = 0 \), the problem is linear and the quadratic formula fails. Technically, linear equations are also quadratics — but so-called degenerate quadratics. This is often a point of contention in the classroom. So, to clarify we should agree that when discussing all quadratics, linear equations are included unless we specify beforehand that \( a \neq 0 \).

Making notation and definitions precise is always a good idea.

Roots vs zeros
This is always a confusion. For any function \( f(x) \) we call the solutions of \( f(x) = 0 \), the roots of the equation. Those same roots are called the zeros of the function. So, equations have roots and functions have zeros.

Memorizing the formula
Students should be encouraged to memorize the quadratic formula to find the roots of the quadratic. This is particularly so for those planning to take additional mathematics courses. This formula is used repeatedly, perhaps more than any other, in all mathematics courses through calculus. Having memorized it, the student will be able to quickly resolve that part of the question at hand.

It is important to note that factoring or solving a quadratic in application is rarely the entire problem at hand. It is usually a part of a larger problem solving mathematical process. Hence skill at this often uninteresting task is at a premium.

Practice and repetition are the best ways to memorize this formula, and one key device to this end is to state it each time it is used. Do it yourself in class and encourage students to do the same. You could even make it as a small component on tests.

Students forget
More than other things students often forget to place the correct signs on the various terms in the quadratic formula. Writing the formula each time it is used helps students remember it correctly. Asking them to write the formula before each application helps even more.

Remark  When the coefficient \( a \neq 1 \), it is best to use the quadratic formula regardless. The alternative, factoring, is an usually a complex task of finding factors in the face of multiple.
Remark One problem students do have when using the quadratic formula is reducing the answers obtained to integers when there are integer solutions. For example application of the quadratic formula to \( x^2 - 4x + 3 \) gives solutions

\[
\frac{4 + \sqrt{16 - 12}}{2}, \quad \text{and} \quad \frac{4 - \sqrt{16 - 12}}{2}
\]

which many students will not simplify to 3 and -1.

An issue of nomenclature

The roots of the quadratic equation \( 2x^2 + 4x + 1 = 0 \), computed from the quadratic formula are \( \pm \frac{1}{2} \sqrt{2} - 1 \), but as decimals are

\[
\frac{1}{2} \sqrt{2} - 1 = -0.2929 \text{ and } -\frac{1}{2} \sqrt{2} - 1 = -1.7071
\]

Since -0.2929 is only an approximation to \( \frac{1}{2} \sqrt{2} - 1 \), we question is whether to use the equality sign or the alternative approximate equality sign \( \doteq \)? There is no apparent universal agreement on this. Of course it is mathematically incorrect to use =, but when it is understood that radicals and the like will always be approximated by decimals, then often the equality sign is used. What is important is that students understand that the decimal is only an approximation of the exact term.

Why is it important to compute the decimal approximations. One reason is so that they can be used in further calculation, and another is to connect the zeros of the function with the roots of the equation as demonstrated graphically.

Remark Here you can use the quadratic grapher to graph the function. This should give a clue to where the roots are.
Factoring
When factoring we write
\[ x^2 + bx + c = (x - r)(x - s) \]

It is important to make clear the signs in the terms \((x - r)(x - s)\). When \(r\) is negative, say \(r = -2\), some students will report the term \((x - 2)\) instead of \((x + 2)\). This is clearly one of the most common errors there are — even at the collegiate level.

Teaching factoring of \(ax^2 + bx + c\)
Many many teachers have their own pet ways of teaching factoring, and of course it’s understandable ... to them. However, there is one method that many teachers use that we will call the X-method. Let’s factor \(x^2 - 4x - 12\). Begin with a large \(X\). Place the coefficient \(b = 4\) on top and the coefficient \(c = 12\) on the bottom of the \(X\). The two factors will be placed at the left and right of the \(X\). Now write down all the factors of \(c\), which in this particular problem is quite a few. Sum them and find the sum that matches \(b\). These are the factors. Place them at the left and right of the \(X\).

<table>
<thead>
<tr>
<th>Factors of (c)</th>
<th>Sum of factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>-12</td>
</tr>
<tr>
<td>-3</td>
<td>4</td>
</tr>
<tr>
<td>-4</td>
<td>3</td>
</tr>
<tr>
<td>-2</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>-6</td>
</tr>
</tbody>
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As you can see the combination of factors that sums to \(-4\) is the last pair, 2 and \(-6\). So, the factorization is \((x - 6)(x + 2)\). Hence the zeros are 6 and \(-2\). Note the signs are reversed.

Arguably, all methods are mathematically equivalent, but the truly nice thing about this one is that it is visually self-organizing with a place for everything, and everything in its place.

**Remark** What happens if there are no two simple factors that add to \(b\)? This means, assuming all coefficients are integers, that the solution involves a radical — and this means use the quadratic formula. Since we have written all the factors of twelve above, it’s easy to see that the similar problem of factoring \(x^2 - 5x - 12\) must involve a radical.

Suppose \(a \neq 1\). How does the X-method work then? In this case the product \(ac\) is placed at the bottom of the \(X\). Follow the same procedure as above and then divide by \(a\) when the numerical factors have been found. For example, find the zeros of \(6x^2 - x - 2\).
Dividing by $a (= 6)$ and reversing the signs, we know then that zeros are $\frac{4}{6} = \frac{2}{3}$ and $-\frac{3}{6} = -\frac{1}{2}$. This procedure then is better at finding the zeros of a quadratic with $a \neq 1$ than factoring it.