Functions

The definition of a function is given as:
A rule of correspondence between two sets such that there is a unique element in the second set assigned to each element in the first set.

The word "unique" is key here. Without it, we merely have a relation. Relations are a larger class of function-like objects, and indeed include functions. They are not unimportant and quite useful, but somewhat more difficult to study. Since functions are so ubiquitous and important to almost every facet of life, we focus on them. The most common method of representing a function is

\[ y = f(x) \]

where \( f \) is the function operation. In that context we have the more mathematical looking definition.

Definition. A function \( y = f(x) \) is a rule that associates with each value of \( x \) in some set \( X \) one and only one value \( y = f(x) \) in some set \( Y \). The set \( X \) is called the domain of the function and the set of values \( f(x) \) for all \( x \) in \( X \) is called the range of the function. Also, \( x \) is called the independent variable, and \( y \) is called the dependent variable. We call \( y \) a function of \( x \).

Which definition students prefer is a matter of taste. The first definition is relatively clear of any sort of terminology and can be understood by anyone knowing only what sets are. But the second definition has with it the assorted trappings and terms needed to discuss mathematical functions. We have the following components all making precise just what is what.

1. The rule - the actual function
2. Domain - what values the rule acts upon
3. Range - what values the rule takes domain elements to
4. Independent variable - the symbol that represents domain elements
5. Dependent variable - the symbol that represents range elements.

Imagine a function as a machine that converts values of \( x \) into values of \( y \). With the new notions of domain and range, we might illustrate this machine as in the figure below
What is also important is that when discussing a function, it is not necessary to have a formula, just the rule, the domain, and the range.

**Domain**

The domain of most functions is usually simple to determine. For real-life problems, though, remember that the associated function may be defined for many more values of the independent variable than physically make sense. For instance, recall the previous examples about the height of a ball.

**Range**

However, the range of a function can be quite difficult to determine. Except for linear and quadratic functions, which are considered later, we need the calculus to determine ranges, and this comes later in the curriculum. In the meantime, try to verify the correctness of the assertions below.

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 2x + 1$</td>
<td>all $x$</td>
<td>all $y$</td>
</tr>
<tr>
<td>$y = \sqrt{x}$</td>
<td>all $x \geq 0$</td>
<td>all $y \geq 0$</td>
</tr>
<tr>
<td>$y = x^2 + 1$</td>
<td>all $x$</td>
<td>all $y \geq 1$</td>
</tr>
<tr>
<td>$y = \frac{1}{x} + 1$</td>
<td>all $x \neq 0$</td>
<td>all $y \neq 1$</td>
</tr>
</tbody>
</table>

**Function examples**

Functions occur naturally in everyday life. Consider, for example, the following functions with time as the independent variable: temperature or barometric pressure at some fixed place, a person’s blood pressure, the Dow-Jones average, and the national debt. Even though no formulas are given, they are functions nonetheless.

**Example** Consider $y = 2x + 1$. Here, $f(x) = 2x + 1$, and for each value of $x$, there is but one value of $y$—namely, $y = 2x + 1$. If $x = 1$, $y = 3$; if $x = -4$, $y = -7$, and so on. Both the domain and range are all real numbers. This function is linear.

**Example** Consider $y = x^2 + 3x + 2$. Here, $f(x) = x^2 + 3x + 2$, and, again, for each value of $x$, there is but one value of $y$—namely, $y = x^2 + 3x + 2$. If $x = 0$, $y = 2$; if $x = 1$, $y = 6$; if $x = -1$, $y = 0$, and so on. The domain is the set of all real numbers. The range is more difficult to
specify; it is the set of all real numbers no smaller than $-\frac{1}{4}$. This function is a quadratic.

**Example**  Here are some functions for which there is no formula.

1. The temperature at a specific point on front step of the State capitol building at any given time. The domain is all time values. The range is all number values, though we could in all practicality limit the range to a reasonable set, say $[-20, 120]$ in Fahrenheit degrees.
2. The height of a person. Here the domain is the set of all people, and the range is the set of all heights.
3. The leader’s current cumulative time in the Tour de France. The domain is any time, and the range is given in hours:minutes:seconds.

**Example**  A ball is thrown straight up from an initial height of 4 ft at an initial velocity of 63 ft/sec. The height at any given time $x$ (in sec) is given by

$$y = 4 + 63x - 16x^2$$

This is a function whose domain is all $x \geq 0$ until the time when the ball touches the ground. The range is all the numbers $y$ (heights) from 0 to the maximum height of the ball. How high is the ball after 1 sec? After 4 sec? After 5 sec? Interpret the answers.

**Solution**  Let $h(x) = 4 + 63x - 16x^2$. Mathematically, the problem asks for the values $h(1)$, $h(4)$ and $h(5)$. They are

$$h(1) = 4 + 63 \cdot 1 - 16 \cdot 1^2 = 51$$
$$h(4) = 4 + 63 \cdot 4 - 16 \cdot 4^2 = 0$$
$$h(5) = 4 + 63 \cdot 5 - 16 \cdot 5^2 = -81$$

So, after 1 sec the ball is 51 ft high, and after 4 sec it is 0 ft high. To explain why the ball is lower after 4 sec, we must conclude that, by the time $x = 4$, the ball has already risen to its highest point (apex) and has fallen to the ground. Now the next value $h(-5) = -81$ seem to indicate that the ball is now 81 feet below the ground. But this is not realistic. What we have here is a situation where the domain of the function is all values from 0 to 4, where the ball returns to the ground. Beyond that the function cannot have a reasonable interpretation. So, technically, we could write

$$h(x) = \begin{cases} 
4 + 63x - 16x^2 & \text{if } 0 \leq x \leq 4 \\
0 & \text{if } 4 < x
\end{cases}$$

For $x \geq 0$, its graph could be expressed as
Domain and Range. It is important to make the following agreement on domain and range: *When the domain of a function is not made explicit, we assume the domain to be every value for which the function is defined or, in functions describing physical quantities, every value for which the function has meaning.* We illustrate this agreement in the following examples.

**Example** Consider $x^2 + y^2 = 1$. As this equation stands, it does not fit the criterion for being a function. That is, it is not a relation of the form $y = f(x)$. We can try to make it into a function by solving for $y$, obtaining

$$y = \pm \sqrt{1 - x^2}$$

Note, though, that when solving for $y$ we must take a square root, and there are always two square roots (of a positive number): a positive one and negative one. So, we obtain two functions: $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$, where, for each value of $x$ ($-1 \leq x \leq 1$), there are two values of $y$. That is, $x^2 + y^2 = 1$ is not a function. For the function $y = \sqrt{1 - x^2}$, the domain is all $x$ with $-1 \leq x \leq 1$, and the range is all numbers $y$ with $0 \leq y \leq 1$. What are the domain and range of $y = -\sqrt{1 - x^2}$?

Definitions are important, as has been long recognized. From the philosopher Thomas Hobbes (1588-1679) we have

*The errors of definitions multiply themselves according as the reckoning proceeds; and lead men into absurdities, which at last they see but cannot avoid, without reckoning anew from the beginning.*

The moral here is to make precise definitions that leave little "wiggle room" for misinterpretations. Mathematicians are keenly aware of this.